Powerfull Nonlinear Plasma Waves from Moderate First Order Perturbations

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1. Abstract

The nonlinear Fourier method of Callebaut consists in concentrating on the family of higher order terms of a single Fourier term of the linearized analysis. Thus we have obtained the higher order terms of plasma perturbations, gravitational ones, etc. In the simplest case of cold plasma this resulted in obtaining an analytical expression for the higher order terms. This allowed to investigate the convergence of the series, which in this case limits the first order amplitude to 1/e of the equilibrium density. For the cases without an analytical expression we developed a numerical-graphical method to obtain the convergence limit. Near this limit the total amplitude of the wave becomes very large. The convergence limit decreases with increasing pressure.

Thus a wave with moderate first order amplitude may carry a very large energy due to the higher orders. Moreover, this energy is concentrated in a very narrow interval of the phase interval $(0, 2\pi)$. This may be relevant in many situations. E.g. in the case of ball lightning a tremendous energy may be accumulated while the glowing is still restricted. The triggering of solar flares or coronal mass ejections may thus be caused. Again, when these eruptions reach the Earth the influence of a first order term may be far too small to cause electric power plants to break down; however, the total of all terms may be much more powerful. Cf. March 1989 when the whole state of Quebec, Canada, was a day without electricity due to a solar storm. This is an alternative mechanism from the one proposed by Callebaut and Tsintsadze based on soliton envelope formation, although there the accent was on the heating of the plasma.

2. Introduction

We have developed a kind of nonlinear Fourier analysis for systems of nonlinear partial differential equations. We have elaborated this in a fair variety of cases: plasma waves, gravitational waves and instabilities, hydrodynamic and magnetohydrodynamic configurations with boundaries or infinite in space [1-4].

Using an analytical or in most cases a numerical - graphical method we were able to derive the convergence ratio, i.e. the maximum amplitude of the initial wave (first order term) to avoid the divergence of the full series of the family of terms associated with a single Fourier term of the linear theory. At the same time it is clear that with a certain initial amplitude and energy a very large amount of energy may be contained in a fairly small volume, which is concentrated over a fraction of the wavelength and moves in space (or oscillates back and forth). This is an attempt to give an answer to the often unexpectedly large energy content of ball lightning (long life in spite of the small luminosity) and to the great variation of lightning balls and their particular features [5]. Moreover we suggest a mechanism based on these concentrated waves to explain the extreme rapidity of energy explosions like solar flares and coronal mass ejections (CMEs) [6-7]. It may be interesting too for certain atmospheric phenomena [8-9].

For simplicity - just to innovate the idea - we work in the case of ball lightning with a rectangular box in which a plasma wave travels back and forth between two reflecting walls. For solar phenomena on the contrary the waves are not bounded and travel away from their birth places.

The plan of the paper is as follows. In section 3 we explain the nonlinear Fourier analysis of Callebaut. In section 4 we give a sketch of the elaboration for plasma waves. In section 5 we explore the application of these concentrated waves to ball lightning. In section 6 we discuss the possible application of those results to solar flares and coronal mass ejections (CMEs). Section 7 contains the conclusions.

3. Essentials of the Nonlinear Fourier Method

To obtain the (linear) dispersion relation one linearizes the system of equations and next applies a Fourier analysis. This has as a result that at least part of the PDEs (partial differential equations) may be “algebraised” and that one may work with one single Fourier term, say $Ae^{i(\omega t+\mathbf{k}\cdot\mathbf{r})}$, in which A is the (arbitrary) amplitude, ω the angular frequency, $\mathbf{k}$ the wave vector, $\mathbf{r}$ the space vector and t the time. (In view of the (linearized) boundary conditions it is often the case that one or two dimensions do not allow a Fourier analysis and that a differential equation has still to be solved, e.g. in the cylindrical case this leads to the inclusion of Bessel functions combined with the Fourier terms for the other directions.) Of course the full linearized solution is then a sum/integral over such terms, still with arbitrary (but very small) amplitudes. Let us call this Fourier analysis the *horizontal* Fourier integral/sum. However, having derived the dispersion relation and sticking to one specific Fourier term with a specific amplitude A, specific ω and specific k, one may look at the higher order terms generated by this specific term by iteration in the nonlinear system: this leads to a series of terms of the type $a_n A^n e^{ni(\omega t+\mathbf{k}\cdot\mathbf{r})}$ as solution; here a_n are the coefficients fixed by substitution in the nonlinear basic system of equations. We may call this the *vertical* Fourier series. In fact it is the normal Fourier series of the function defined by the nonlinear system and by its first term $Ae^{i(\omega t+\mathbf{k}\cdot\mathbf{r})}$. Actually, this function is fully fixed as a solution of the system of equations and its first Fourier term which in a sense acts as a kind of initial condition, or rather as the "linear approximation condition".

Generalization to several "initial terms"

Suppose now that we consider two "initial terms", taken from the horizontal solutions: $A_1 e^{i(\omega_1 t+\mathbf{k}_1\cdot\mathbf{r})}$ and $A_2 e^{i(\omega_2 t+\mathbf{k}_2\cdot\mathbf{r})}$ where the pairs $(\omega_1,\mathbf{k}_1)$ and $(\omega_2,\mathbf{k}_2)$ are incommensurable (meaning that $\chi_1\equiv\omega_1 t+\mathbf{k}_1\cdot\mathbf{r}$ and $\chi_2\equiv\omega_2 t+\mathbf{k}_2\cdot\mathbf{r}$ do not satisfy a relation of the type $n\chi_1=m\chi_2$ with n and m integers, otherwise the approach is somewhat different).

Substituting their sum in the system of equations leads to three parts: a vertical Fourier series corresponding to $A_1e^{i\chi_1}$, a similar vertical Fourier series corresponding to $A_2e^{i\chi_2}$ and a mixed series. However, once the vertical Fourier series is obtained for one specific $Ae^{i\chi}$ these three series are written down at once using combinatorial coefficients for each order.

The same is true when considering several "initial terms". Clearly once we consider mixing (interference) of initial terms we are working with an authentic nonlinear solution and this method may justly be called a kind of *nonlinear Fourier analysis*.

4. Sketch of the Analysis for Plasma Waves

Here we give a sketch of the elaboration of the method to the case of plasma waves in an infinite homogeneous medium. We neglect gravity, viscosity, resistivity and the magnetic contributions. We consider here the cold plasma case only: the ions are immobile. The basic equations are then respectively the equation of continuity, motion, Poisson, and polytropics:

$$\partial_t n + div(n\mathbf{v}) = 0, \tag{1}$$

$$nm(\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}) = -divp + e\nabla\varphi, \tag{2}$$

$$\Delta\varphi = e(n-n_0)/\varepsilon, \tag{3}$$

$$p = Kn^\Gamma, \tag{4}$$

where n is the number density of the electrons, n_0 their equilibrium density, $\mathbf{v}$ their velocity, φ is the electrical potential, p their pressure, e and m are the electron charge and mass, ε is the permittivity, whose value in vacuum is 8.85×10^{-12} C/Vm, K and Γ (polytropic exponent) are constants.

The linear perturbation is expressed as a Fourier series. We fix one term say $A\exp[i\chi]$ as mentioned above. We thus develop only one family of higher order terms corresponding to a single Fourier term of the linearized analysis. The nonlinear terms generate then $a_2A^2\exp[3i\chi]$, $a_3A^3\exp[3i\chi]$, etc. with a_2, a_3, $\cdots$, coefficients to be determined.

Using $\chi=\omega t+\mathbf{k}\cdot\mathbf{r}$ as a single variable reduces the system to ordinary differential equations:

$$\omega n_-' + \mathbf{k}\cdot(n_-\mathbf{v}_-)' = 0, \tag{5}$$

$$m_-(\omega+\mathbf{v}_-\cdot\mathbf{k})\mathbf{v}_-' = e\mathbf{k}\varphi' - \mathbf{k}p', \tag{6}$$

$$k^2\varphi'' = \frac{e}{\varepsilon}(n_- - n_0), \tag{7}$$

$$p = Kn^\Gamma, \tag{8}$$

where the accent means the derivative with respect to χ. Integrating the continuity equation (5) we obtain

$$(\omega + \mathbf{k}\cdot\mathbf{v}_-)n_- = \in_- = \omega n_0. \tag{9}$$

This is then used in reducing the system of equations (5) - (8) to a differential equation of second order:

$$\left[\frac{(\Gamma-2)k^2 v_{s-}^2 n^{\Gamma+1}}{n_0^{\Gamma+1}} + 3\omega^2\right] n'^{\,2} + \left(\frac{k^2 v_{s-}^2 n^{\Gamma+1}}{n_0^{\Gamma+1}} - \omega^2\right) n n'' = \frac{\omega_p^2 n^4 (n-n_0)}{n_0^3}, \tag{10}$$

where n_0 is the equilibrium density, $k^2 v_{s-} = K_-\Gamma n_0^{\Gamma-1}/m$ is the sound velocity of electrons and $\omega_p^2 = (e^2 n_0)/(m\varepsilon)$ is the square of the electron plasma frequency. We calculated a number of coefficients numerically using mathematica and then inferred the analytic expression. We obtained e.g. for the cold plasma case

$$n = n_0\left(1 + \sum_{j=1}^{N} \frac{j^j}{j!} A^j e^{ij\chi}\right),\ \mathbf{v} = \frac{\mathbf{k}\omega}{k^2}\sum_{j=1}^{N}\frac{j^{j-1}}{j!}A^j e^{ij\chi},\ \varphi = \frac{en_0}{k^2\varepsilon}\sum_{j=1}^{N}\frac{j^{j-2}}{j!}A^j e^{ij\chi}. \tag{11}$$

We integrated as well equation (10) to obtain a differential equation of first order and even to obtain a fully integrated equation: the same results, e.g. expressions (11), were obtained, but the times needed for the three procedures to calculate the coefficients was variable and strongly dependent on any simplification or additional effect taken into account (e.g. when the ions are mobile as well). The series (11) is convergent provided $A < e^{-1}$ ($e \approx 2.71828\cdots$) i.e. if the linearized perturbed density has an amplitude larger than 37% of the equilibrium density n_0, the series is no more convergent.

A graphical method confirmed this: Indeed, summing up N terms of the density, n, and plotting the result for χ in the interval 0, 2π (or even 0, π) yields an oscillating graph. If there was any value in this interval for n less than zero the series has to be rejected since the electron number density may not become negative. It turns out that this graphical method confirms the radius of convergence found analytically rather well; we performed the summation in some cases up to $N = 7000$ to verify the result accurately. Actually when we exceeded the limit of convergence for A slightly the number of terms might be rather limited (say ten or twenty), except very close to the limit of convergence. This numerical/graphical method turned out a powerful tool in all cases where we did not have a systematic analytical expression for the coefficients. Moreover, it showed that near the limit of convergence all terms combined in a narrow interval (i.e. much smaller than (0,2π)) of the phase to form together a concentrated high energy wave.

5. Application to Ball Lightning

We elaborate the following model: consider a rectangular box which contains plasma, created e.g. by a lightning flash and in which a plasma wave travels back and forth between two opposite sides. Clearly the use of a rectangular box is just for mathematical reasons to be able to apply the results derived above. The reflecting walls are, however, a serious problem. Maybe the surface of the box has a higher density than inside the box so that the plasma frequency at the walls is too high for the waves to penetrate the walls, just making them bounce back again and again. This may explain the sometimes sudden explosion of the lightning balls just like soap bubbles. This high dense plasma surface (or maybe chasma ("charged plasma") surface as the quasi - neutrality may not be granted in this thin surface layer) may also show some elasticity allowing some deformations according to the surfaces. However, this problem can be totally avoided by developing the theory for a sphere, with waves running round.

The full plasma wave (sum of all the Fourier terms) may contain moderate energy if the amplitude of the linear approximation is small as compared to the radius of convergence. It may contain a very large amount of energy when the amplitude of the first order term is close to the radius of convergence. Moreover, this energy is concentrated over a small fraction of the space covered by a wavelength or multiples of it (i.e. from one side of the rectangular box to the other), as mentioned above.

Hence the heating locally is not continuous: the peak of the wave comes and goes and passes by again. This intermittent local heating (and cooling) may explain a low average heat and low light emission, in spite of the great energy content.

The well-known dispersion relation for plasma waves:

$$\omega^2 = \omega_p^2 + k^2 v_s^2 \tag{12}$$

where ω is the angular frequency, ω_p is the plasma frequency and v_s is the sound velocity, shows that the frequencies are somewhat above the plasma frequency. They should not be too much above this plasma frequency if the idea of a surface with a somewhat higher density (hence higher ω'_p) is correct so that $\omega'_p > \sqrt{\omega_p^2 + k^2 v_s^2}$, preventing the penetration. Let us now concentrate on k. Lightning balls have dimensions roughly between 0.01 and 0.2 m, requiring the box, hence the wavelength (or the multiple) to be of the same order. With a sound speed of 330 m/s (it may be higher if e.g. the temperature is higher and because the small mass of the electrons inside the box) this requires k to be correspond to the acoustic frequencies of 10 to 1000 kHz. This is plausible as indeed thundering has frequencies in the range and above those heard by humans. These acoustic waves may be either generators of the perturbations corresponding to the plasma wave dispersion relation given above and/or they may be consequences.

6. Application to Solar Flares and CMEs

Such a wave may be the trigger of solar flares and coronal mass ejections (CMEs). These are huge outburst of energy going to several times 10^{25} joule or roughly more than a million of heavy hydrogen bombs or one tenth of the energy emitted by the Sun in one second. Such a wave may be generated by some smaller phenomena like bright points on the solar surface. Those smaller phenomena or explosions still eject huge amounts of energy of various kinds. Suppose a wave of the kind described above is thus generated. It may travel and reach a region in which an enormous quantity of magnetic energy is stored which is on the verge of instability. (Note by the way the large energy density caused by moderate magnetic fields: with an induction of 0.01 tesla (100 gauss) the energy density is $B^2/2\mu = 500 J/m^3$. Moreover the volumes can extend over hundreds, even thousands of kilometers.) By its great and concentrated energy the wave may trigger the instability. In particular its concentration of energy in a small strip may cause turbulence and hence change the resistivity from its normal value to its turbulent one, which, according to some calculations, may be even five orders of magnitude larger. As the wave moves on it enlarges the narrow strip where the resistivity is in the turbulent regime and thus allows the fast conversion of the magnetic field into heat and eruptive phenomena. Of course the wave soon gets exhausted (unless the interaction with the magnetic field feeds it) but with a strip of some extend the whole magnetic region may be "set to fire". It should be noted that this turbulence causing a much enhanced resistivity is often invoked as a means of explaining the short duration of the flash phase of e.g. a solar flare, which is of the order of 1000 seconds, although the whole phenomenon may take a day. With normal resistivity the dissipation of the magnetic energy would take a year or more. It has to be added that the filamentary structure of the magnetic field allows smaller time scales. The characteristic decay time is proportional to L^2/η where L is a typical scale length of the field and η is the resistivity. Hence, using e.g. a length which is 10 times smaller in view of the filamentary structure shortens the decay times by a factor 100. Probably the combined effect of filaments and enhanced resistivity is required to explain the rapid conversion of magnetic energy into other forms.

7. Conclusion

Although their formation and constitution may be different these concentrated waves resemble solitary waves. Cf. Callebaut and Tsintsadze [6-7] and may produce similar effects. Thus the sudden instability of solar flares, of CMEs and of filament bands encircling a domain of the solar surface or the whole Sun may be explained by a triggering effect. These concentrated waves allow the enhancement of the resistivity by turbulence. Similarly, when the outburst of solar flares or CMEs reach the Earth they may be accompanied by such concentrated waves and cause all kinds of damage by triggering instabilities in electric power plants and instruments, e.g. in air planes.

It is suggested that these waves may be created by lightning generating ball lightning and then are transformed in peaked waves which run around in the ball. Yet, several improvements are required. E.g. an analysis of these waves in spherical or cylindrical coordinates, an investigation of their duration, etc.

REFERENCES

1. Callebaut, D. K., "Lineaire en Niet-lineaire Perturbaties in Hydro-, Magneto- en Gravitodynamica," *Simon Stevin 45, Ghent, Belgium*, 1-315, 1972.
2. Callebaut, D. K. and G. K. Karugila, "Nonlinear Fourier Analysis for Unmagnetized Plasma Waves," *Physica Scripta*, Vol. 68, No. 1, 7 - 21, 2003.

3. Callebaut, D. K. and G. K. Karugila, "Nonlinear Fourier Analysis for Unmagnetized Multi-species Plasma Waves," *International Journal of Theoretical Physics*, Submitted.
4. Callebaut, D. K., G. K. Karugila and H. Kikuchi, "Nonlinear Stability of a Gravitating Medium with Cosmological Constant," *Kuwait J. Sci. Eng.*, Vol. 31, No. 2, 33-45, 2004.
5. Callebaut, D. K. and G. K. Karugila, "Nonlinear Fourier Approach yielding High Fields in Ball Lightning," *Eighth International symposium on Ball Lightning (ISBL04), 3rd-6th August 2004, National Central University (NCU), Chung-Li, Taiwan.*, 38-43, 2004.
6. Callebaut, D. K. and N. L. Tsintsadze, "Linear and Nonlinear Interaction of Electromagnetic Waves with Weakly Ionized Magnetized Plasmas," *Physica Scripta*, Vol. 52, No. 5, 565-571, 1995.
7. Callebaut, D. K. and N. L. Tsintsadze, "Heating of Plasma by Alfvn Waves Envelope," *Physica Scripta*, Vol. 50, No. 3, 283-289. 1994.
8. Kikuchi, H., in *Handbook of Atmospheric Electrodynamics*, Vol. 1, Ed. H. Volland, CRC press, Bonn, 167, 1995.
9. Kikuchi, H., "Electrodynamics in Dusty and Dirty Plasmas (Gravito-Electrodynamics and EHD)," *ASSL*, Vol. 258, Kluwer Acad. Press, Dordrecht, 2001.
10. Dijkhuis G., (ed.), *Sixth International Symposium on Ball Lightning (ISBL99)*, August 1999, UIA (now CDE), University of Antwerp, B-2610 Antwerp, Belgium, 1999.
11. Veselovsky I. S., "Non-local mass ejection development with and without magnetic reconections," in *Multi-Wavelength Investigations of Solar Activity*, Proc. of IAUS 223, A. V. Stepanov, E. E. Benevolenskaya and A. G. Kosovichev (eds.), Cambridge University Press, G. B., 517, 2004.
12. Webb D. F., "Coronal Mass Ejections and Space Weather," in *Multi-Wavelength Investigations of Solar Activity*, Proc. of IAUS 223, A. V. Stepanov, E. E. Benevolenskaya and A. G. Kosovichev (eds.), Cambridge University Press, G. B., 449, 2004.