On Vladimirov's Approximation for Ideal Inhomogeneous MHD

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Abstract

Vladimirov and Vladimirov and Moffat have considered configurations in ideal magnetohydrodynamics, i.e. inviscid and perfectly conducting. The matter is considered as incompressible. However, the density is allowed to vary slowly. They base the following approximation on this slow variation: they omit the mass density in front of the total derivative of the velocity in the equation of motion. Normally the mass density should appear in front of Du. This is a tremendous simplification which allows them to obtain various interesting results concerning the stability of the configurations. However, in such a kind of approximation the results might be only crude. However, in many applications the results are OK, because crucial in those papers is the vanishing of $\nabla\rho \times \nabla\varphi$. Often both gradients are parallel and the results obtained by Vladimirovs approximation are nevertheless valid, e.g. in the application to inhomogeneous gas clouds and protostars. Moreover for small density gradients and/or nearly parallel gradients the approximation is fair. We even suggest an approximation which may be more correct and avoids the term $\nabla\rho \times \nabla\varphi$. Hence for linear perturbations and stability analyses the results may turn out to be acceptable. However, for nonlinear stability a more extended analysis is required.

1. Introduction

Vladimirov [1] and Vladimirov and Moffat [2-4]have considered configurations in ideal magnethydrodynamics (MHD), i.e. inviscid and perfectly conducting. The matter is considered as incompressible. However, the density is allowed to vary slowly, hence they use what they call a Boussinesq approximation. The basic equations according to Vladimirovs approximation are:

$$\partial_t \mathbf{h} = curl(\mathbf{u} \times \mathbf{h}) \tag{1}$$

$$D\mathbf{u} \equiv (\partial_t + \mathbf{u}.\nabla)\mathbf{u} = -\nabla p - \mathbf{j} \times \mathbf{h} - \rho\nabla\varphi \tag{2}$$

$$D\rho = 0 \tag{3}$$

$$\nabla.\mathbf{u} = 0 \tag{4}$$

$$\nabla.\mathbf{h} = 0 \tag{5}$$

Where $\mathbf{u}$ is the velocity field, p the pressure, $\mathbf{h}$ the magnetic field, $\mathbf{j} = curl\,\mathbf{h}$ the current density and φ the gravitational potential. Eq.(1) gives, according to the MHD approximation, the evolution of the magnetic field due to the dynamo term while the resistive term is neglected. Eq.(3) expresses the incompressibility. Eq.(4) is the equation of continuity in the incompressible case. Eq.(5) expresses the conservation of magnetic flux.

Our concern is here mainly the equation of motion, eq.(2). Normally this expresses the conservation of momentum or momentum density: ρ should appear in front of Du in it. The correct left hand side of eq.(2) may be written

$$\rho D\mathbf{u} = (\partial_t + \mathbf{u}.\nabla)(\rho\mathbf{u}) = (\partial_t + \mathbf{u}.\nabla)\mathbf{U} \tag{6}$$

in view of eq.(3) and putting $\mathbf{U} \equiv \rho u$. However, eq.(2) is used while eq.(6) shows a mixing up of u (in the total derivative) and U on which the total derivative acts. Alternatively we may divide out ρ so that it does not occur anymore in front of $D\mathbf{u}$, but then other consequences have to be considered as we will suggest below.

In fact some of the present authors (A.H.K and D.K.C) have used Vladimirov's approximation with success in several papers [5-7]. Hence a critical analysis may be useful.

2. Analysis

2.1. Use of Reduced Quantities

We start with a somewhat semantic remark concerning the terminology "Boussinesq approximation". Chandrasekhar [8] gives a detailed analysis of the Boussinesq approximation and shows that indeed one may neglect the variations due to thermal expansion of the mass density, except in the term containing the gravitational potential. Chandrasekhar shows that for the case of thermal stability the Boussinesq approximation is not even required in the linear perturbation analysis. Hence ρ is treated there as not varying in the inertia term. However, in the studies at hand [1-7], the variation in ρ is a variation in space in the configuration, besides possible variations due to thermal effects or to a perturbation (which moreover should be negligible for incompressible matter). Hence we are not dealing with a proper Boussinesq approximation, but with a different kind, although there is some similarity. We prefer to call it Vladimirov's approximation.

In fact a quantitative analysis should be performed to estimate the influence of neglecting $grad\rho$ and the error induced by Vladimirovs approximation. Here we give a qualitative argument. For a perturbation of an equilibrium u is a quantity of first order and usually its derivatives are too. Replacing ρ by a constant (an average ρ_{av}, say) in front of $D\mathbf{u}$ neglects the terms of second order in $\rho D\mathbf{u}$ (although there are two kinds of first order involved, which are not directly related: one for the variation in ρ due to its inhomogeneity and one for the variation in $\mathbf{u}$, due to the perturbation). Still ρ_{av} should be in front of $D\mathbf{u}$, unless the whole equation is divided by ρ_{av}. The quantities in the right hand side of eq.(2) then become reduced or normalized quantities: p is reduced to p/ρ_{av}, $\mathbf{h}$ is reduced to $\mathbf{h}/\sqrt{\rho_{av}}$, j is reduced to $j/\sqrt{\rho_{av}}$, which is consistent with eq.(1). In the last term in eq.(2) we may drop ρ altogether, which is more correct than writing $(\rho/\rho_{av})\nabla\varphi$. This further simplifies eq.(2) and avoids the inconsistencies between u and U. Moreover, a crucial aspect in the investigations by Vladimirov and co-authors [1-4] and by Khater and co-authors [5-7] is the vanishing of the expression $\nabla\rho \times \nabla\rho$. In fact in the ideal situation of a perfectly conducting fluid the magnetic field is frozen in the fluid:

$$L\mathbf{h} \equiv \partial_t \mathbf{h} - \nabla \times (\mathbf{u} \times \mathbf{h}) = 0 \tag{7}$$

where L is a kind of Lie derivative. Next with the vorticity field $\mathbf{w} \equiv \nabla \times \mathbf{u}$ one obtains from eq.(2) that

$$L\mathbf{w} = \nabla \times (\mathbf{j} \times \mathbf{h}) - \nabla\rho \times \nabla\varphi \tag{8}$$

Let $\mathbf{g}(\mathbf{r}, \mathrm{t})$ be an arbitrary solenoidal field satisfying

$$\nabla \times (g \times h) = 0 \tag{9}$$

and $\mathbf{m}(\mathbf{r}, \mathrm{t})$ be defined by

$$L\mathbf{m} = \mathbf{j} + \mathbf{g} \tag{10}$$

$$\nabla.\mathbf{m} = 0 \tag{11}$$

Then one obtains with $\mathbf{W} \equiv \mathbf{w} + \nabla \times (\mathbf{h} \times \mathbf{m})$

$$L\mathbf{W} = -\nabla\rho \times \nabla\varphi \tag{12}$$

When the right hand side vanishes the expression $\mathbf{W}$ yields the appropriate frozen-in field generalization of $\mathbf{w}$ for ideal MHD. (Note that, since $\mathbf{h}$ and $\mathbf{W}$ are two independent frozen-in fields, it follows that $\nabla \times (\mathbf{h} \times \mathbf{W})$ is a frozen-in field too, and by iteration, an infinite family of such derived frozen-in fields may be constructed.) Next one constructs a general Casimir functional as an integral of an arbitrary function of both conserved fields. Using this Casimir functional a linear stability criterion is obtained and applied to magnetized gas clouds [7].

Leaving out ρ in the gravitational term gets rid of the term $\nabla\rho \times \nabla\varphi$ and the analyses made are valid in first approximation. Moreover, it may be added that even when this term appears the two gradients are often parallel or at least nearly parallel as e.g. in stars and magnetized gas clouds. This is the case of the papers [5-7] where the configurations under consideration are spherically symmetric or nearly so as the gravitational force turns out to be dominant.

Note by the way that we do not have to calculate ρ_{av}: we just use ρ to make the reduction and allow its spatial variation in the final results. This seems even a better approximation as only the terms containing explicitly $\nabla\rho$ are neglected.

2.2. Consistency between Equations with u and U

The use of reduced quantities seems a reasonable way out for the problem at hand. Yet we consider whether the use of $\mathbf{u}$ and $\mathbf{U}$ may be useful for an alternative approach. We consider the consistency between the basic equations. It was suggested in the introduction that eq.(6) may replace the right hand side of eq.(2) and that then two quantities $\mathbf{u}$ and $\mathbf{U}$ occur. However, considering only a linearized perturbation theory leaves us with $\mathbf{U}$ alone in eq.(6). However, eq.(4) and eq.(1) still deal with $\mathbf{u}$. They can hardly be converted into equations with $\mathbf{U}$ unless we fall back on the use of the reduced quantities in the previous subsection. In fact in the linearized version it is simpler to stick to $\mathbf{u}$ only, which is quite well possible. However, in the further elaboration following eq.(2) we will still face the term $\nabla\rho \times \nabla\varphi$ which has to vanish and which is crucial for the developments in references [1-7].

The conservation of energy should use $\int \rho u^2 d\tau$ or $\int (U^2/\rho) d\tau$ for its kinetic part. With the reduced quantities as explained above this may be quite reasonable, although we are then speaking of a reduced energy in which the non kinetic contributions are divided by ρ. Although $\mathbf{u}$ now occurs quadratically the division is by ρ itself, not its square and the approximation will not be worse than in the basic equation. Hence the equation for conservation of energy still seems usable to the same degree of approximation as the linearized theory.

3. Conclusion

Vladimirovs approximation is not a proper Boussinesq approximation, although it is usually called as such because of some similarities. It allows a great simplification with usable results at least for the equilibrium configurations and for the first order stability analysis by using a kind of normalized or reduced quantities. A quantitative measure for the approximation is still lacking. Qualitatively the gradient of ρ should not exceed a few percent. Along another line of thought we made an attempt to use $\mathbf{U} = \rho\mathbf{u}$ (momentum density) instead of $\mathbf{u}$ (velocity) as basic quantity. However, the consistent elaboration of this becomes very involved and not really suitable in general.

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