

# Adaptive techniques for Landau–Lifshitz–Gilbert equation with magnetostriction

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## Abstract

In this paper we propose a time–space adaptive method for micromagnetic problems with magnetostriction. The considered model consists of coupled Maxwell's, Landau–Lifshitz–Gilbert (LLG) and elastodynamic equations. The time discretization of Maxwell's equations and the elastodynamic equation is done by backward Euler method, the space discretization is based on Whitney edge elements and linear finite elements, respectively. The fully discrete LLG equation reduces to an ordinary differential equation, which is solved by an explicit method, that conserves the norm of the magnetization.

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## 1. Introduction

Ferromagnetic materials are used in large variety of devices, such as magnetic sensors, actuators, reading–writing heads, information storage media, passive circuit elements, etc. The understanding of magnetic processes in these materials is essential for their use in the magnetic industry. Magnetoelastic coupling causes the deformation of the materials when subject to magnetic field, and changes of magnetization when subject to stress. The interactions between the magnetic and mechanical properties of the materials, which are also called magnetostriction, were described e.g., in [6,9].

We consider a time interval  $(0, T)$  together with a convex polygonal domain  $\Omega \subset \mathbb{R}^3$  with a boundary  $\Gamma$ , which consists of two non-overlapping parts  $\Gamma_D$  and  $\Gamma_N$ . Further we set  $Q_T = (0, T) \times \Omega$ . We denote in bold the standard vectorial function spaces like  $\mathbf{L}_2(\Omega)$ ,  $\mathbf{H}^1(\Omega)$ ,  $\mathbf{H}(\text{curl}, \Omega)$ . We will use the following spaces of test functions:

$$\mathbf{V} = \{\boldsymbol{\varphi} \in \mathbf{H}^1(\Omega), \boldsymbol{\varphi} = \mathbf{0} \text{ on } \Gamma_D\}$$

and

$$\mathbf{W} = \{\boldsymbol{\psi} \in \mathbf{H}(\text{curl}, \Omega), \mathbf{v} \times \boldsymbol{\psi} = \mathbf{0} \text{ on } \Gamma\}.$$

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We will use the notation  $\| \cdot \|$ ,  $\| \cdot \|_1$  and  $\| \cdot \|_{\text{curl}}$  for usual norms in  $L_2(\Omega)$ ,  $\mathbf{H}^1(\Omega)$  and  $\mathbf{H}(\text{curl}, \Omega)$ , respectively.

The evolution of magnetization is governed by the Landau–Lifshitz–Gilbert (LLG) equation

$$\partial_t \mathbf{M} = \frac{|\gamma|}{1 + \alpha^2} \left( \mathbf{H}_T \times \mathbf{M} + \alpha \frac{\mathbf{M}}{|\mathbf{M}|} \times (\mathbf{H}_T \times \mathbf{M}) \right), \tag{1.1}$$

where  $\alpha$  is a damping constant and  $\gamma$  is the gyromagnetic factor. The magnetization has a prescribed modulus  $|\mathbf{M}| = M_s$  and variable orientation. The vector  $\mathbf{H}_T$  represents the total magnetic field in the ferromagnet

$$\mathbf{H}_T = \mathbf{H} + \mathbf{H}_a + \mathbf{H}_m, \tag{1.2}$$

where  $\mathbf{H}$  is the magnetic field,  $\mathbf{H}_m$  is the magnetostrictive component and  $\mathbf{H}_a = K P(\mathbf{M})$  describes the anisotropy. We have neglected the exchange field in (1.2), which is possible for some applications. For mathematical analysis of the LLG equations see e.g. [18].

The constant  $K$  is a constant characterizing the anisotropy of the material. We discuss the case of a ferromagnetic crystal with one distinguished axis, which is the axis of easiest magnetization represented by a unit vector  $\mathbf{p}$ ,  $|\mathbf{p}| = 1$ . The symbol  $P(\mathbf{M})$  denotes the projection of  $\mathbf{M}$  on  $\mathbf{p}$ , i.e.,

$$P(\mathbf{M}) = (\mathbf{p} \cdot \mathbf{M})\mathbf{p}. \tag{1.3}$$

The magnetic field  $\mathbf{H}$  is obtained from Maxwell’s equations. Here we consider Maxwell’s equations in a simplified form, the so-called eddy current equation

$$\mu_0 \partial_t \mathbf{H} + \nabla \times \frac{1}{\sigma} \nabla \times \mathbf{H} = -\mu_0 \partial_t \mathbf{M}, \tag{1.4}$$

where  $\sigma$  is a conductivity constant and  $\mu_0$  the permeability of vacuum. For simplicity, we consider following boundary conditions for (1.4):

$$\mathbf{v} \times \mathbf{H} = \mathbf{0}. \tag{1.5}$$

Micromagnetic models with eddy current without magnetostriction equation were studied in e.g., [17,16,10].

The component  $\mathbf{H}_m = \left\{ \sum_{ijk} \lambda_{ijkl}^m \sigma_{ij} M_k \right\}$  describes the magnetostriction. We assume a linear dependence of the stress tensor  $\boldsymbol{\sigma} = \{\sigma_{ij}\}$  on the elastic part of the total strain  $\boldsymbol{\epsilon}^e = \{\epsilon_{kl}^e\}$ , which is the inverse form of Hook’s linear law

$$\sigma_{ij} = \sum_{kl} \lambda_{ijkl}^e \epsilon_{kl}^e. \tag{1.6}$$

The total strain is defined as

$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \tag{1.7}$$

where  $\mathbf{u} = \{u_{ij}\}$  is the displacement vector.

The magnetostrictive component of the total strain has the form

$$\epsilon_{ij}^m(\mathbf{M}) = \sum_{kl} \lambda_{ijkl}^m M_k M_l. \tag{1.8}$$

The total strain is then given by

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^e + \boldsymbol{\epsilon}^m. \tag{1.9}$$

The tensors  $\lambda^e$  and  $\lambda^m$  are symmetric ( $\lambda_{ijkl} = \lambda_{jikl} = \lambda_{ijlk} = \lambda_{klij}$ ) and positive definite

$$\sum_{ijkl} \lambda_{ijkl} \xi_{ij} \xi_{kl} \geq \lambda^* \sum_{ij} \xi_{ij}^2 \tag{1.10}$$

with bounded components.

The first term in the right-hand side of (1.1) causes a precession of  $\mathbf{M}$  around  $\mathbf{H}_T$  and it is not dissipative, while the second term is dissipative. A scalar multiplication of (1.1) by  $\mathbf{M}$  gives

$$\partial_t \mathbf{M} \cdot \mathbf{M} = \frac{1}{2} \partial_t |\mathbf{M}|^2 = 0. \tag{1.11}$$

The time integration shows that the length of  $\mathbf{M}$  remains constant at any time  $t > 0$ ,

$$|\mathbf{M}(t)| = |\mathbf{M}(0)| = |\mathbf{M}_0|. \tag{1.12}$$

The stress tensor  $\boldsymbol{\sigma}$  and displacement  $\mathbf{u}$  (assuming zero external forces) satisfy the conservation of momentum equation

$$\rho \partial_{tt} \mathbf{u} - \nabla \cdot \boldsymbol{\sigma} = \mathbf{0}, \tag{1.13}$$

where we assume the mass density  $\rho$  to be constant independent on the deformation.

We take the boundary conditions

$$\begin{aligned} \boldsymbol{\sigma}_i \cdot \mathbf{v} &= 0 \quad \text{on } \Gamma_N, \quad i = 1, 2, 3, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma_D \end{aligned} \tag{1.14}$$

and initial data ( $\mathbf{x} \in \Omega$ ),

$$\begin{aligned} \mathbf{u}(0, \mathbf{x}) &= \mathbf{u}_0(\mathbf{x}), \\ \partial_t \mathbf{u}(0, \mathbf{x}) &= \mathbf{v}_0(\mathbf{x}). \end{aligned} \tag{1.15}$$

The symbol  $\mathbf{v}$  stands for the outward unit normal vector on the boundary.

The following variational formulation of (1.13) can be obtained using the boundary conditions (1.14), Hook’s relation (1.6), the definition of the total strain (1.9) and the symmetry of the tensors  $\boldsymbol{\lambda}^e, \boldsymbol{\lambda}^m$ ,

$$(\rho \partial_{tt} \mathbf{u}, \boldsymbol{\varphi}) + (\boldsymbol{\lambda}^e \boldsymbol{\epsilon}(\mathbf{u}), \boldsymbol{\epsilon}(\boldsymbol{\varphi})) = (\boldsymbol{\lambda}^e \boldsymbol{\epsilon}^m(\mathbf{M}), \boldsymbol{\epsilon}(\boldsymbol{\varphi})) \quad \forall \boldsymbol{\varphi} \in \mathbf{V}. \tag{1.16}$$

The variational formulation of (1.4) reads as

$$\mu_0 (\partial_t \mathbf{H}, \boldsymbol{\psi}) + \frac{1}{\sigma} (\nabla \times \mathbf{H}, \nabla \times \boldsymbol{\psi}) = -\mu_0 (\partial_t \mathbf{M}, \boldsymbol{\psi}) \quad \forall \boldsymbol{\psi} \in \mathbf{W}. \tag{1.17}$$

## 2. Numerical scheme

In the following text we normalize all physical constants without loss of generality.

We approximate the  $\mathbf{u}$  in space by the piecewise linear finite elements and  $\mathbf{H}$  is discretized by lowest order Whitney edge elements. The approximation of the LLG equation in time is based on the method presented in [15]. Eqs. (1.16), (1.17) were discretized in time by the backward Euler method.

The time interval  $[0, T]$  we divide into  $n$  equidistant subintervals  $[t_{i-1}, t_i]$ , where the time step  $\tau_i = t_i - t_{i-1}$ . Further we denote

$$\mathbf{z}_i = \mathbf{z}(t_i), \quad \delta \mathbf{z}_i = \frac{\mathbf{z}_i - \mathbf{z}_{i-1}}{\tau_i}, \quad \delta^2 \mathbf{z}_i = \frac{\delta \mathbf{z}_i - \delta \mathbf{z}_{i-1}}{\tau_i} \tag{2.1}$$

for any vector function  $\mathbf{z}$ .

For each time  $t_i, 0 \leq i \leq n$ , we introduce a family of regular triangulations  $\mathcal{T}_i$ . For each triangle  $K \in \mathcal{T}_i$ , let  $h_K$  stand for its diameter. We also denote by  $\mathcal{E}_i$  the set of all edges  $e$  from  $\mathcal{T}_i$ ,  $h_e$  denotes the size of  $e \in \mathcal{E}_i$ . The discrete finite element space for approximation of the displacement  $\mathbf{u}$  read as

$$\mathbf{V}^h = \{ \boldsymbol{\varphi}^h \in \mathbf{V}, \forall K \in \mathcal{T}_i, \boldsymbol{\varphi}^h|_K \in \mathcal{P}_1(K) \}, \tag{2.2}$$

where  $\mathcal{P}_1(K)$  is the space of linear affine functions on  $K$ .

The magnetic field  $\mathbf{H}$  will be approximated by Whitney edge elements. We define the polynomial space

$$R_K = \{\phi, \phi(\mathbf{x}) = \mathbf{a}_k + \mathbf{b}_K \times \mathbf{x}, \mathbf{a}_K, \mathbf{b}_K \in \mathbb{R}^3\}. \tag{2.3}$$

Then we define the space of lowest order Whitney edge elements  $\mathbf{W}^h \subset \mathbf{W}$  as

$$\mathbf{W}^h = \{\psi \in \mathbf{H}(\text{curl}, \Omega), \psi|_K \in R_K\}. \tag{2.4}$$

The degrees of freedom on  $\mathbf{W}^h$  are located at edges of the mesh

$$\Sigma_K = \left\{ \int_e \mathbf{u} \cdot \boldsymbol{\tau}, e \subset K, \boldsymbol{\tau} \text{ is unit tangent vector to } e \right\}. \tag{2.5}$$

From the previous definition of degrees of freedom we can construct an interpolation operator  $r^h : \mathbf{H}^2 \rightarrow \mathbf{W}^h$ , with the property

$$\|\mathbf{u} - r^h \mathbf{u}\|_{H(\text{curl}, \Omega)} \leq Ch \|\mathbf{u}\|_{H^2(\Omega)}. \tag{2.6}$$

The approximate solution to  $\mathbf{M}, \mathbf{u}, \boldsymbol{\sigma}, \mathbf{H}$  are denoted by  $\mathbf{m}^h, \mathbf{u}^h, \boldsymbol{\sigma}^h, \mathbf{h}^h$ , respectively. We take the following approximation of initial data:

$$\begin{aligned} \mathbf{M}_0^h &= I^h \mathbf{M}_0, \\ \mathbf{u}_0^h &= I^h \mathbf{u}_0, \\ \mathbf{v}_0^h &= I^h \mathbf{v}_0, \\ \boldsymbol{\sigma}_0^h &= \lambda^e \in (I^h \mathbf{u}_0), \\ \mathbf{h}_0^h &= r^h \mathbf{H}_0. \end{aligned} \tag{2.7}$$

The method that we use for the coupled system (1.1), (1.13) was introduced in [2]. On each time interval  $[t_{i-1}, t_i]$ ,  $i = 1, \dots, n$ , instead of the nonlinear equation (1.1), we consider a linear differential equation of the form

$$\partial_t \mathbf{m}^h = (\mathbf{H}_T(\mathbf{h}_{i-1}^h, \mathbf{m}_{i-1}^h, \boldsymbol{\sigma}_{i-1}^h)) \times \mathbf{m}^h + \frac{\mathbf{m}^h}{|\mathbf{m}_{i-1}^h|} \times ((\mathbf{H}_T(\mathbf{h}_{i-1}^h, \mathbf{m}_{i-1}^h, \boldsymbol{\sigma}_{i-1}^h)) \times \mathbf{m}_{i-1}^h), \tag{2.8}$$

which can be written as

$$\begin{aligned} \partial_t \mathbf{m}^h &= \left[ (\mathbf{H}_T(\mathbf{h}_{i-1}^h, \mathbf{m}_{i-1}^h, \boldsymbol{\sigma}_{i-1}^h)) - (\mathbf{H}_T(\mathbf{h}_{i-1}^h, \mathbf{m}_{i-1}^h, \boldsymbol{\sigma}_{i-1}^h)) \times \frac{\mathbf{m}_{i-1}^h}{|\mathbf{m}_{i-1}^h|} \right] \times \mathbf{m}^h \\ &= \mathbf{a}_{i-1} \times \mathbf{m}^h. \end{aligned} \tag{2.9}$$

We use the fact that the equation (where  $\mathbf{a}$  is a constant vector in time)

$$\begin{aligned} \partial_t \mathbf{m} &= \mathbf{a} \times \mathbf{m}, \\ \mathbf{m}(0) &= \mathbf{m}_0 \end{aligned} \tag{2.10}$$

has exact solution of the form

$$\mathbf{m}(t) = \mathbf{m}_0^\parallel + \mathbf{m}_0^\perp \cos(|\mathbf{a}|t) + \frac{\mathbf{a}}{|\mathbf{a}|} \times \mathbf{m}_0^\perp \sin(|\mathbf{a}|t), \tag{2.11}$$

where  $\mathbf{m}_0 = \mathbf{m}_0^\parallel + \mathbf{m}_0^\perp$ ,  $\mathbf{m}_0^\parallel$  is parallel to  $\mathbf{a}$  and  $\mathbf{m}_0^\perp$  is perpendicular to  $\mathbf{a}$  (see [15]).

Since  $\mathbf{a}_{i-1}$  is constant on  $[t_{i-1}, t_i]$ , from (2.10), (2.11) we easily obtain the solution of (2.9) on  $[t_{i-1}, t_i]$  for  $i = 1, \dots, n$ .

Note that this scheme conserves pointwise the modulus of  $\mathbf{m}^h$ , i.e.:

$$|\mathbf{m}^h(t)| = |\mathbf{m}_{i-1}^h| \quad \text{for } t \in [t_{i-1}, t_i], \quad i = 1, \dots, n. \tag{2.12}$$

Further, we can write the discrete version of (1.16) for  $i = 1, \dots, n$  as

$$(\delta^2 \mathbf{u}_i^h, \psi) + (\lambda^e \epsilon(\mathbf{u}_i^h), \epsilon(\psi)) = (\lambda^e \epsilon^m(\mathbf{m}_i^h), \epsilon(\psi)) \quad \forall \psi \in \mathbf{V}^h, \tag{2.13}$$

here we have taken  $\delta \mathbf{u}_0^h = \mathbf{v}^h(0) = I^h \mathbf{v}_0$ .

The existence of  $\mathbf{u}_i^h \in \mathbf{V}^h$  in (2.13) for any  $i = 1, \dots, n$  is guaranteed by the Lax–Milgram theorem and Korn’s inequality (cf. [5, Theorem 9.2.16]).

Finally, the fully discrete version of (1.17) read as

$$(\delta \mathbf{h}_i^h, \psi) + (\nabla \times \mathbf{h}_i^h, \nabla \times \psi) = -(\delta \mathbf{m}_i^h, \psi) \quad \forall \psi \in \mathbf{W}^h. \tag{2.14}$$

The existence of  $\mathbf{h}_i^h \in \mathbf{W}^h$  is guaranteed by the Lax–Milgram theorem.

Note, that Eqs. (2.9), (2.13) and (2.14) are linear and decoupled, therefore they can be solved separately by computing elementwise the exact solution (2.11) and a solving a linear system arising from (2.13), (2.14) on every time step, respectively.

### 3. A posteriori error estimate

We define piecewise linear interpolations of the discrete solutions as

$$\mathbf{m}_n(t) = \mathbf{m}_{i-1}^h + \frac{(t - t_{i-1})}{\tau_i} (\mathbf{m}_i^h - \mathbf{m}_{i-1}^h), \tag{3.1}$$

$$\mathbf{u}_n(t) = \mathbf{u}_{i-1}^h + \frac{(t - t_{i-1})}{\tau_i} (\mathbf{u}_i^h - \mathbf{u}_{i-1}^h), \tag{3.2}$$

$$\mathbf{v}_n(t) = \mathbf{v}_{i-1}^h + \frac{(t - t_{i-1})}{\tau_i} (\mathbf{v}_i^h - \mathbf{v}_{i-1}^h), \tag{3.3}$$

$$\mathbf{h}_n(t) = \mathbf{h}_{i-1}^h + \frac{(t - t_{i-1})}{\tau_i} (\mathbf{h}_i^h - \mathbf{h}_{i-1}^h), \tag{3.4}$$

where  $t \in (t_{i-1}, t_i)$  and

$$\begin{aligned} \mathbf{v}_0^h &= \mathbf{v}_0, \\ \mathbf{v}_i^h &= \frac{\mathbf{u}_i^h - \mathbf{u}_{i-1}^h}{\tau_i}, \\ \mathbf{h}_0^h &= I^h \mathbf{H}_0. \end{aligned} \tag{3.5}$$

Then (2.13) can be reformulated as (cf. [4])

$$\begin{aligned} (\partial_t \mathbf{u}_n(t), \boldsymbol{\varphi}) - (\mathbf{v}_n(t), \boldsymbol{\varphi}) &= \mathbf{0}, \\ (\partial_t \mathbf{v}_n(t), \boldsymbol{\varphi}) + (\lambda^e \epsilon(\mathbf{u}_n(t)), \epsilon(\boldsymbol{\varphi})) &= (\lambda^e \epsilon^m(\overline{\mathbf{m}}_n(t)), \epsilon(\boldsymbol{\varphi})), \end{aligned} \tag{3.6}$$

where  $\overline{\mathbf{m}}_n(t) = \mathbf{m}_i$  for  $t \in (t_{i-1}, t_i)$ .

Eq. (2.14) is equivalent to

$$(\partial_t \mathbf{h}_n(t), \psi) + (\nabla \times \mathbf{h}_n(t), \nabla \times \psi) = -(\partial_t \mathbf{m}_n(t), \psi) \quad \forall \psi \in \mathbf{W}^h. \tag{3.7}$$

for  $i = 1, \dots, n, t \in (0, T)$ .

We define the following error indicators:

$$\begin{aligned} \gamma_i^\tau &= \|\nabla \times \mathbf{h}_i^h - \nabla \times \mathbf{h}_{i-1}^h\|^2, \\ \gamma_i^h &= \sum_{K \in \mathcal{F}_i} h_K^2 \|\delta \mathbf{h}_i^h + \delta \mathbf{m}_i^h\|_{L_2(K)}^2 + \sum_{e \in \mathcal{E}_i} h_e \|\mathbf{v}_e \times \nabla \times \mathbf{h}_i^h\|_{L_2(e)}^2 \\ &\quad + \sum_{K \in \mathcal{F}_i} h_K^2 \|\nabla \cdot \delta \mathbf{h}_i^h\|_{L_2(K)}^2 + \sum_{e \in \mathcal{E}_i} h_e \|\mathbf{v}_e \cdot (\delta \mathbf{h}_i^h + \delta \mathbf{m}_i^h)\|_{L_2(e)}^2, \end{aligned}$$

$$\begin{aligned} \mu_i^\tau &= \|\lambda^e(\epsilon(\mathbf{u}_i^h) - \epsilon(\mathbf{u}_{i-1}^h))\|^2 + \|\delta \mathbf{v}_i^h\|^2 + \|\mathbf{h}_i^h - \mathbf{h}_{i-1}^h\|^2 + \|\mathbf{m}_i^h - \mathbf{m}_{i-1}^h\|^2, \\ \mu_i^h &= \sum_{K \in \mathcal{T}_i} \left\| \frac{h_K}{\tau_i} \delta \mathbf{v}_i^h \right\|_{L_2(K)}^2 + \sum_{e \in \mathcal{E}_i} h_e \|[ \mathbf{v}_e \cdot \boldsymbol{\lambda}^e \epsilon(\mathbf{u}_i^h) ]_e \|_{L_2(e)}^2 + \sum_{e \in \mathcal{E}_i} h_e \|[ \mathbf{v}_e \cdot \boldsymbol{\lambda}^e \epsilon^m(\mathbf{m}_i^h) ]_e \|_{L_2(e)}^2. \end{aligned} \tag{3.8}$$

The terms that contain  $i - 2$  vanish for  $i = 1$ . Here  $[\cdot]_e$  denotes the jump along the edge  $e$ ,  $\mathbf{v}_e$  is the unit outward normal vector to the triangle  $K$  on  $e \in \partial K$ .

**Remark 3.1.** We expect the a posteriori error estimate to hold for  $1 \leq i \leq n$

$$\begin{aligned} &\|\mathbf{u}(t_i) - \mathbf{u}_n(t_i)\|^2 + \|\mathbf{v}(t_i) - \mathbf{v}_n(t_i)\|_{-1}^2 + \|\mathbf{M}(t_i) - \mathbf{m}_n(t_i)\|^2 \\ &+ \|\mathbf{H}(t_i) - \mathbf{h}_n(t_i)\|^2 + \int_0^{t_i} \|\nabla \times (\mathbf{H}(s) - \mathbf{h}_n(s))\|^2 ds \\ &\leq \|\mathbf{v}_0 - \mathbf{v}_0^h\|_{-1}^2 + \|\mathbf{u}_0 - \mathbf{u}_0^h\|^2 + \|\mathbf{M}_0 - \mathbf{m}_0^h\|^2 + \|\mathbf{H}_0 - \mathbf{h}_0^h\|^2 \\ &+ C \sum_{j=1}^i \tau_j (\gamma_j^\tau + \gamma_j^h + \mu_j^\tau + \mu_j^h). \end{aligned} \tag{3.9}$$

**Remark 3.2.** The proof of the previous a posteriori error estimate is beyond the scope of this paper and will be given elsewhere. The idea of the proof is as follows. The error indicators  $\gamma^h, \gamma^\tau$  represent the error caused by the discretization of the eddy current equation. They can be obtained by using the approach for time-dependent parabolic equations from [13] with, e.g., the work [3,11], where the a posteriori error of space discretization of eddy current equations is derived. The proof can be concluded by combining the previous result with the approach from [1], where the estimates of the same type as  $\mu^h, \mu^\tau$  were obtained for the system (2.9), (2.13) without eddy current equation.

#### 4. Adaptive algorithm

A number of time–space adaptive strategies have been described e.g., in [7,8,12], etc. For adaptive strategies in the context of numerical micromagnetics see [14].

Here, we split the error indicators into two parts. The time error is controlled by  $\gamma_i^\tau$  and  $\mu_i^\tau$ . The expression

$$\gamma_i^h + \mu_i^h = \sum_{K \in \mathcal{T}_i} \gamma_{i,K}^h + \mu_{i,K}^h \tag{4.1}$$

is used to control the error caused by spatial discretization.

The proposed adaptive algorithm can be described as follows. For a given tolerance  $TOL$  start with  $\mathcal{T}_0, \tau_0, \mathbf{m}_0, \mathbf{u}_0, \mathbf{h}_0$ :

1. until  $t_{i-1} < t_n$  set  $\tau_i = \tau_{i-1}, \mathcal{T}_i = \mathcal{T}_{i-1}$ ;
2. set  $t_i = t_{i-1} + \tau_i$  and compute the discrete solution, if  $\gamma_i^\tau + \mu_i^\tau \leq \varepsilon_\tau^r TOL$  proceed with the space refinement step 3, else decrease  $\tau_i$  step and repeat step 1;
3. for all  $K \in \mathcal{T}_i$ , if  $\gamma_{i,K}^h + \mu_{i,K}^h > \varepsilon_h^r TOL/N$  mark  $K$  for refinement, if  $\gamma_{i,K}^h + \mu_{i,K}^h < \varepsilon_h^c TOL/N_i$  mark  $K$  for coarsening;
4. refine/coarsen mesh and compute new solution, if  $\gamma_i^\tau + \mu_i^\tau \leq \varepsilon_\tau^c TOL$  increase  $\tau_i$  and go to step 2 (this can be repeated several times, otherwise we proceed to the next time step with, i.e., we go to step 1).

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